Доклады Болгарской академии наук
Comptes rendus de l'Académie bulgare des Sciences
Tome 29, N° 3, 1976

MATHÉMATIQUES
Analyse fonctionelle

A BOUNDARY VALUE PROBLEM IN LINEAR SPACES

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(Submitted on August 12, 1975)

A compact, convex subset K of a locally convex linear space is called a topological simplex if every continuous real-valued function on the set $E(K)$ of the extreme points of K admits a continuous, affine continuation in K.

It is well-known that K is a topological simplex iff it is a simplex in the sense of Choquet and the set of its extreme points is closed (and therefore compact).

We start by stating the following theorem, which we shall use later.

Theorem 1. If K is a topological simplex, then every continuous map of the set $E(K)$ of the extreme points of K into a convex, compact subset of a locally convex linear space admits a continuous, affine continuation in K.

Proof. Let $\psi(p)$ be a continuous mapping of $E(K)$ into a compact, convex subset N of a linear locally convex space Y and let $g(y)$ be a continuous linear real-valued functional in Y. Then $g(\psi(p))$ is a continuous real-valued function on $E(K)$. Let $\theta_g(x)$ be its continuous, affine continuation in K and let L be the set of the points x of K, for which the equation

$$\theta_g(x)=g(y)$$

admits a solution y in N for every g, which does not depend on g.

Since $E(K)\subset L$ holds, we find $K\subset L$. On the other hand, N is compact and therefore y is a continuous function of x. This function is the desired continuation of $\psi(p)$.

Definition. Let m be an affine, continuous mapping from K into K and let φ be a continuous real-valued, affine function in K. The couple (m, φ) is said to be a decomposing pair if in K

$$0\leq\varphi(x)\leq 1,$$
$$m(x)\in\varphi(x)K,$$
$$x-m(x)\in(1-\varphi(x))K$$

holds.

The points p and q are said to be separated by a decomposing pair (m, φ) if $\varphi(p)\neq\varphi(q)$ holds.

Theorem 2. A compact, convex set K in a locally convex linear space is a topological simplex iff the decomposing pairs separate the extreme points of K.

The proof calls for several lemmas.

Lemma 1. Let K be a convex and compact set and (m, φ) be a decomposing pair in K. If s is an extreme point of K and μ is a real number such that $0 \leqq \mu \leqq 1$ holds, then the set of points x of K satisfying the equation

$$m(x) = \mu x + (\varphi(x) - \mu)s \tag{1}$$

is an extreme subset of K.

Proof. We put

$$l(x) = m(x) - \varphi(x)s$$

and

$$n(x) = l(l(x) + s) - 2\mu l(x) + \mu^2(x - s) + s.$$

If p is an extreme point of K, then $m(p) = \varphi(p)p$ holds. Thus $l(p) = \varphi(p)(p - s)$.

Since $n(x) \in K$ holds for the extreme points of K, it holds for every x in K. Since s is an extreme point of K, the set A of the points x of K satisfying $n(x) = s$ is an extreme subset of K. On the other hand, this set coincides with the set B of the points x satisfying (1). Indeed, since the implication $A \supset B$ is trivial, we need only consider the implication $B \supset A$. Now observe that the extreme points of A belong to B and this completes the proof.

Lemma 2. If the decomposing pairs (m, φ) separate the extreme points of the convex, compact set K, the set $E(K)$ of the extreme points of K is closed (and therefore compact).

Proof. Let $\{p_\alpha\}$ be a directed system of extreme points of K converging to p_0. We denote by $F_{m,\varphi,s}$ the set of the points x of K satisfying the equality

$$m(x) - \varphi(x)s = \varphi(p_0)(x - s),$$

where s is an extreme point of K. By lemma 1 the set $F_{m,\varphi,s}$ is an extreme subset of K. Clearly, $p_0 \in F_{m,\varphi,s}$ since $m(p_0) = \varphi(p_0)p_0$. In order to prove that p_0 is an extreme point of K, we shall show that the intersection F of the extreme sets $F_{m,\varphi,s}$ has one single extreme point, which therefore coincides with p_0. This extreme point of F is an extreme point of K as well, since F is an extreme subset of K. Suppose there are two different extreme points p_1 and p_2. Since the decomposing pairs separate the extreme points, there is a decomposing pair (m, φ) such that $\varphi(p_1) \neq \varphi(p_2)$ holds. We may suppose that $\varphi(p_1) \neq \varphi(p_0)$. Clearly, F is an extreme subset of K. Hence p_1 is an extreme point of K and therefore $m(p_1) = \varphi(p_1)p_1$. On the other hand,

$$m(p_1) - \varphi(p_1)s = \varphi(p_0)(p_1 - s).$$

Hence

$$\varphi(p_1)(p_1 - s) = \varphi(p_0)(p_1 - s),$$

which is a contradiction, since we may choose $s = p_2$.

Lemma 3. Let (m_1, φ_1) and (m_2, φ_2) be two decomposing pairs. Then there is such a decomposing pair (m, φ) that for the extreme points p of K

$$\varphi(p) = \varphi_1(p)\varphi_2(p)$$

holds.

Proof. Let s be an extreme point of K. We write down

$$l_1(x) = m_1(x) - \varphi_1(x)s,$$
$$l_2(x) = m_2(x) - \varphi_2(x)s.$$

Then the couple of the functions

$$m(x) = l_2(l_1(x) + s) + \varphi(x)s,$$
$$\varphi(x) = \varphi_2(l_1(x) + s) - (1 - \varphi_1(x))\varphi_2(s)$$

is a decomposing pair with the desired property, since for the extreme points p of K

$$l_1(p)=\varphi_1(p)(p-s),\quad l_2(p)=\varphi_2(p)(p-s),$$
$$\varphi(p)=\varphi_1(p)\varphi_2(p),$$
$$m(p)=\varphi_1(p)\varphi_2(p)p$$

holds.

Proof of theorem 2. Let K be a convex, compact subset of a locally compact linear space. Suppose that the decomposing pairs separate the extreme points of K. We shall prove that every continuous real-valued function $f(p)$ on the set $E(K)$ of the extreme points of K admits a continuous, affine continuation on K. By lemma 2 the set $E(K)$ is compact. Now consider the linear hull R of the real-valued affine, continuous functions $\varphi(x)$ on K such that $m(x)$ can be chosen for the couple (m, φ) to be a decomposing pair. By lemma 3 the linear space R is an algebra on the set $E(K)$ of the extreme points of K. Hence by the Stone-Weierstrass theorem we can find a function $f_n(x)$ in R satisfying

$$|f(p)-f_n(p)|<\frac{1}{n},\quad (n=1, 2, 3, \ldots)$$

on $E(K)$. Hence

$$|f_m(p)-f_n(p)|<\frac{1}{n}+\frac{1}{m},\quad (n,\ m=1, 2, 3, \ldots).$$

On the other hand, the functions of R are continuous and affine and therefore

$$|f_n(x)-f_m(x)|<\frac{1}{n}+\frac{1}{m}$$

holds in K, as well. Clearly, $\lim_{n\to\infty} f_n(x)$ is the desired continuation of $f(p)$.

Now we shall prove the converse. Suppose that every continuons function on the set of the extreme points of K admits a continuous affine continuation on K. Let p_1 and p_2 be two different extreme points. Let $f(x)$ be a continuous linear functional which separates p_1 and p_2. Since $f(x)$ is bounded on K, we may choose the constants α and β so as to have $0<\alpha f(x)+\beta<1$ on K. On the other hand, the function $(\alpha f(p)+\beta)p$ is a continuous map of the set $E(K)$ of the extreme points p of K into the compact and convex set $\bigcup_{0\leq\lambda\leq 1}\lambda K$. Hence, by theorem 1 this map admits a continuous, affine continuation $m(x)$ on K. The couple $(m(x), \alpha f(x)+\beta)$ is then a decomposing pair which separates the points p_1 and p_2.

Corollary. The set $E(K)$ of the extreme points of a topological simplex K is compact.

Proof. By theorem 2 the decomposing pairs separate the extreme points of K. Hence by lemma 2 the set $E(K)$ is compact.

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